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ON THE SPECTRA AND INVERTIBILITY OF CERTAIN OPERATORS IN SYSTEMS THEORY

Ciprian Foias *

Allen Tannenbaum †

Department of Mathematics *
Indiana University
Bloomington, Indiana 47405

Department of Electrical Engineering †
University of Minnesota
Minneapolis, Minnesota 55455

Department of Mathematics †
Ben-Gurion University of the Negev
Beer Sheva, Israel

1. Introduction

In this paper we study the following kind of problem. Let T be an *arbitrary* contraction. We will define a certain function which will lead to an explicit procedure for the computation of the isolated spectrum of $w(T)w(T)^*$, T any contraction. We will then show how this generalizes the results of [1-3]. (See (2.1) and (2.2) below.) We will moreover apply our theory to operators $T \in C_{.0}$ and in particular derive a relatively simple expression for the calculation of the discrete spectrum of $w(T)$.

We should note that from the control engineering point of view, this means that one can now solve an important special case of the general model-matching problem discussed in [5] for multivariate distributed systems as a well as treat a whole new range of Hankel norm approximation problems. We plan to return to these topics in a future applications oriented paper.

2. Main Theorem

In this section we will formulate and prove our main result on the computation of the isolated points of the spectrum for the class of operators that we discussed in the Introduction. In order to do this we will first have to set up some notation.

Accordingly, we will let T denote an arbitrary contraction on a separable Hilbert space H. We will let $w \in H^\infty$ be a rational function which we express as $w = p/q$ where p and q are relatively prime polynomials with $n = \max\{\deg p, \deg q\}$. We will assume that w is not a constant times a Blaschke product.

Set

$$R := \mathbf{R}^+ \setminus \sigma(w(T)w(T)^*)$$

$$P_\rho := q(T)\left[1 - \frac{1}{\rho^2} w(T)w(T)^*\right] q(T)^*$$

$$= q(T)q(T)^* - \frac{1}{\rho^2} p(T)p(T)^*.$$

Note that P_ρ^{-1} exists for $\rho \in R$.

We now define the following crucial function of $\rho \in R$:

$$\nu(\rho) := \text{spectral radius of } (I - T^n T^{*n}) P_\rho^{-1} .$$

It is standard (see for example [6]) that in point of fact

$$\nu(\rho) = \| D_{T^{*n}} P_\rho^{-1} D_{T^{*n}} \|.$$

(For A a contraction, we set $D_A := (I - A^* A)^{1/2}$.)

We are now ready to state the main result of this paper:

THEOREM (2.1) Notation as above. Let $\bar{\rho} \in$ boundary R and suppose that

$$\bar{\rho} \notin \{ |w(z)| : z \in \partial D \cap \sigma(T) \}.$$

Then $\lim_{\rho \to \bar{\rho}} \nu(\rho) = +\infty$ for $\rho \in R$.

Proof. Suppose to the contrary that $\lim_{\rho \to \bar{\rho}} \nu(\rho) < \infty$, for $\rho \in R$. Then clearly there exists a sequence $\rho_j \in R$, $\rho_j \to \bar{\rho}$, such that $\nu(\rho_j) \le M < \infty$, that is we have $\| P_{\rho_j}^{-1/2} D_{T^{*n}} \| \le \sqrt{M}$ for all j. Without loss of generality we may assume that

$$P_{\rho_j}^{-1/2} D_{T^{*n}} \to L \text{ weakly,} \tag{1}$$

$$P_{\rho_j}^{1/2} \to P_{\bar{\rho}}^{1/2} \text{ in norm.} \tag{2}$$

Hence using (1) and (2) we see that

$$\langle P_{\bar{\rho}}^{1/2} Lh, k \rangle - \langle D_{T^{*n}} h, k \rangle = \langle Lh, P_{\bar{\rho}}^{1/2} k \rangle - \langle D_{T^{*n}} h, k \rangle =$$

$$\langle Lh, P_{\bar{\rho}}^{1/2} k \rangle - \langle P_{\rho_j}^{-1/2} D_{T^{*n}} h, P_{\rho_j}^{1/2} k \rangle \to 0 \text{ as } j \to \infty.$$

Thus we have

$$P_{\bar{\rho}}^{1/2} L = D_{T^{*n}}. \tag{3}$$

Next since $0 \in \sigma(P_{\bar{\rho}}^{1/2})$ (because $\bar{\rho} \in \sigma(w(T)w(T)^*)$), we see that there exists a sequence $x_j \in H$, $\|x_j\| = 1$ such that

$$P_{\bar{\rho}}^{1/2} x_j \to 0 \text{ strongly as } j \to \infty. \tag{4}$$

But then from (3) we get that for any $z \in H$

$$|\langle D_{T^{*n}} x_j, z \rangle| = |\langle P_{\bar{\rho}}^{1/2} x_j, Lz \rangle| \tag{5}$$

$$\le \|P_{\bar{\rho}}^{1/2} x_j\| \, \|Lz\|$$

$$\le \|P_{\bar{\rho}}^{1/2} x_j\| \, \|L\| \, \|z\|.$$

In particular for $z = D_{T^{*n}} x_j$, we get

$$\|D_{T^{*n}} x_j\| \le \|L\| \, \|P_{\bar{\rho}}^{1/2} x_j\| \to 0 \text{ as } j \to \infty$$

which implies that

$$\|T^{*n} x_j\| \to 1 \text{ as } j \to \infty. \tag{6}$$

Set $y_j := T^{*n} x_j$. We then note that for $k = 0, \ldots, n$,

$$\|T^{*k} x_j - T^{n-k} y_j\|^2 \le 2 - 2\|y_j\|^2 \to 0 \text{ as } j \to \infty. \tag{7}$$

Now we let

$$\psi(z) := z^{2n} \left(q(z)\overline{q(1/\bar{z})} - \frac{1}{\bar{\rho}^2} p(z)\overline{p(1/\bar{z})} \right).$$

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Then from (4) and (6) we get that $\psi(T)y_j \to 0$. Since we have assumed that $w(z)$ is not a constant times a Blaschke product, it is easy to see that $\psi(z)$ is not identically 0. Moreover we have that deg $\psi(z) \le 2n$.

Let $\lambda_1, \ldots, \lambda_m$ denote the zeros of ψ in the open unit disc D. Note that $m \le n$. Since $\psi(z)$ is such that

$$\frac{\psi(z)}{z^{2n}} = \overline{\psi(1/\overline{z})}$$

we can write $\psi(z) = \psi_1(z)\psi_2(z)$ where

$$\psi_1(z) = \prod_{i=1}^{m}(z - \lambda_i)$$

$$\psi_2(z) = \prod_{i=1}^{m}(z - \frac{1}{\overline{\lambda}_i}) \cdot \prod_{j=1}^{k}(z - \zeta_j)$$

where $|\zeta_1| = \cdots = |\zeta_k| = 1$. (Note that if $\lambda_i = 0$ for some i, then $1/\overline{\lambda}_i = \infty$ and hence this factor will not appear in $\psi_2(z)$).

By hypothesis $\overline{\rho} \notin \{|w(z)| : z \in D \cap \sigma(T)\}$. Hence $\zeta_1, \cdots, \zeta_k \notin \sigma(T)$. Moreover since $\psi(T)y_j \to 0$ as $j \to \infty$ and $\psi_1(T) = \psi_2(T)^{-1}\psi(T)$, we conclude

$$(T - \lambda_1) \cdot \cdots \cdot (T - \lambda_m)y_j \to 0. \tag{8}$$

(Note that form (6) and (7) $\|y_j\| \to 1$ and $\|T^n y_j\| \to 1$ as $j \to \infty$.)

Define for each $j = 1, 2, \cdots$, a complex vector space H_j spanned by $y_j, \cdots, T^{m-1}y_j$, and set

$$T_j = P_{H_j}|H_j$$

where $P_{H_j} : H \to H_j$ denotes projection. Then if

$$\psi_1(z) = z^m + a_1 z^{m-1} + \cdots + a_m$$

we have from (8) that

$$T^m y_j = -(a_1 T^{m-1}y_j + \cdots + a_m y_j) + z_j \tag{9}$$

where $\|z_j\| \to 0$ as $j \to \infty$.

But

$$\|TT^{m-1}y_j - T_jT^{m-1}y_j\| = \|(I - P_j)T^m y_j\| \le \|z_j\| \to 0 \text{ as } j \to \infty.$$

Therefore

$$\|T_j^m y_j + a_1 T_j^{m-1}y_j + \cdots + a_m y_j\| = \tag{10}$$

$$\|T_jT^{m-1}y_j + a_1T^{m-1}y_j + a_m y_j\|$$

$$\le \|\psi_1(T)y_j\| + \|z_j\| = 2\|z_j\| \to 0 \text{ as } j \to \infty.$$

Moreover since

$$\|T_j^m y_j - T^m y_j\| \le \|z_j\| \to 0$$

and $m \le n$ we infer that

$$\|T_j^m y_j\| \to 1 \text{ as } j \to \infty. \tag{11}$$

Now from (10) and (11) and the fact that that there exist unitary operators $V_j : H_j \to \mathbf{C}^N$ for some fixed $N \le m$, we see

$$V_jT_jV_j^{-1} \to T_\infty,\ V_jy_j \to y_\infty \text{ strongly} \tag{12}$$

where

$$\psi_1(T_\infty)y_\infty = 0 \text{ and} \tag{13}$$

$$\|y_\infty\| = 1 = \|T_\infty^m\ y_\infty\|. \tag{14}$$

But we have at long last arrived at the desired contradiction. Indeed (14) implies that T_∞ is an isometry on a finite dimensional vector space. On the other hand (13) implies that T_∞ has an eigenvalue of modulus < 1. This contradiction finishes the proof of the theorem.

□

We now want to derive a special case of (2.1) for a class of operators which appear most frequently in engineering applications. Recall that we say a contraction T is of class $C_0(n)$ if there exists $u \in H^\infty$, u not identically 0 such that $u(T) = 0$ and the defect indices of T equal n. (See [7] for a detailed discussion about this important class of contractions.)

Then we have :

COROLLARY (2.2) Set for $\rho \in R$

$$\vartheta(\rho) := \text{ trace of } (I - T^nT^{*n})P_\rho^{-1}.$$

Then for $\overline{\rho}$ as in (2.1) we have $\lim_{\rho \to \overline{\rho}} \vartheta(\rho) = +\infty$.

Proof. Immediate from (2.1).

□

REMARK (2.3) In [1] we proved (2.2) for $T \in C_0(1)$. In the next section we will show how easy it is to apply our techiques to a typical kind of operator which arises in control theory.

3. Application to Control Theoretic Example

In this section we would like to illustrate via a nontrivial example that Theorem (2.1) gives an explicit method for computing the discrete spectrum of those operators of current interest in engineering. The interested reader should compare the computations given in this section to the ones given in [2] and [3] for the compressed shift.

Let $T: H \to H$ denote a contraction. Then following [7] we say that T is of class $C_{\cdot 0}$ if $T^{*n}h \to 0$ as $n \to \infty$ for all $h \in H$. Now let $U: K \to K$ denote the minimal isometric dilation of T, and set

$$L := \overline{(U - T)H}$$

$$L_* := \overline{(I - UT^*)H}\ .$$

It is easy to see that L, L_* are subspaces of K which are wandering for U. (See [7].)

It is a basic fact ([7]) that T admits a functional model such that

$$K \cong H^2(L_*), \quad U \cong S \tag{15}$$

$$H \cong H^2(L_*) \ominus \Theta H^2(L), \quad T \cong P_H S \mid H$$

where S denotes the unilateral shift, $P_H: K \to H$ projection, and $\Theta: H^2(L) \to H^2(L_*)$ is an inner operator-valued function. (Notice that "$\cong$" denotes "is unitarily equivalent to".) In what follows we will always identify T with its functional model via (15).

Using the notation of Section 2, we want to derive an explicit formula for $\nu(\rho)$ in case

$$w(z) = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \alpha\delta - \beta\gamma \neq 0 ,$$

and for $T \in C_{\cdot 0}$. The reader will see that an identical procedure can be carried out in principle for any rational $w(z)$.

We will always assume that $\rho \in R$. Then

$$P_\rho = [(\gamma T + \delta)(\bar{\gamma} T^* + \bar{\delta}) - \frac{1}{\rho^2}(\alpha T + \beta)(\bar{\alpha} T^* + \bar{\beta})]$$

$$= (A + BT + \bar{B}T^* + CTT^*)$$

where

$$A := |\delta|^2 - \left[\frac{1}{\rho^2}\right]|\beta|^2$$

$$B := \gamma\bar{\delta} - \left[\frac{1}{\rho^2}\right]\alpha\bar{\beta}$$

$$C := |\gamma|^2 - \left[\frac{1}{\rho^2}\right]|\alpha|^2 .$$

We now want to compute an explicit formula for

$$\nu(\rho) = \|(I - TT^*)^{1/2} P_\rho^{-1} (I - TT^*)^{1/2}\|.$$

In order to do this in this general setting we will first have to define a unitary equivalence $\phi_*: L_* \to \mathbf{D}_{T^*}$ where $\mathbf{D}_{T^*} := \overline{(I - TT^*)^{1/2} H}$. Indeed for $l_* = (I - UT^*)h \in L_*$, we set $\phi_* l_* := (I - TT^*)^{1/2} h$. (It is easy to check that ϕ_* is unitary; see [7].) Notice then that

$$(I - TT^*)^{1/2} P_\rho^{-1} (I - TT^*)^{1/2} = phi_* (I - UT^*) P_\rho^{-1} \phi_* (I - UT^*)$$

and consequently

$$\nu(\rho) = \|(I - UT^*) P_\rho^{-1} \phi_* (I - UT^*)\|. \tag{16}$$

Next one can check that for any $f \in H$

$$(I - UT^*) f = f(0) =: f_0. \tag{17}$$

Then we set for given $h \in H$

$$\mu := \mu(h) := P_\rho^{-1} \phi_* (I - UT^*) h . \tag{18}$$

But clearly from (16), $\nu(\rho)$ may be defined as the norm of the operator from $\mathbf{D}_{T^*} \to L_*$ given by $\phi_* h_0 \to \mu_0$. We will now give an explicit formula for this operator.

Using (18) we have that

$$(A + BT + \bar{B}T^* + CTT^*)\mu = \phi_* h_0. \tag{19}$$

Now

$$T^* = \bar{z}(\mu - \mu_0) \tag{20}$$

$$T\mu = z\mu - \Theta\mu_{-1} \tag{21}$$

where

$$\Theta^* \mu = \mu_{-1}\bar{z} + \text{(higher order terms in } \bar{z}).$$

Moreover

$$(I - TT^*)\mu = (I - \Theta\Theta_0^*)\mu_0 \tag{22}$$

for $\Theta_0 := \Theta(0)$. Setting $F := A + C$, from (19), (20), (21) we get

$$[C(I - \Theta\Theta_0^*) + \bar{B}\,\bar{z}\,]\mu_0 + B\Theta\mu_{-1} + \phi_* h_0 \tag{23}$$

$$= (F + Bz + \bar{B}\,\bar{z}\,)\mu.$$

Multiplying (23) by z we have

$$[Cz(I - \Theta\Theta_0^*) + \bar{B}]\mu_0 + B\Theta z\mu_{-1} \tag{24}$$

$$= (Bz^2 + Fz + \bar{B})\mu - (\phi_* h_0)z.$$

Note that even though this relationship has been derived on the unit circle, since all of the functions admit an analytic continuation to all of D, we can regard (24) as valid on all of D.

Next applying Θ_* to (23) and multiplying by z, we get

$$[Cz(\Theta^* - \Theta_0^*) + \bar{B}\Theta^*]\mu_0 + Bz\mu_{-1} \tag{25}$$

$$= (Bz^2 + Fz + \bar{B})\Theta^*\mu - \Theta^*(\phi_* h_0)z.$$

Note that we can analytically extend Θ^* to the complement of the unit disc by setting $\Theta^*(z) := \Theta(1/\bar{z})^*$ for z such that $|z| > 1$. Hence even though (25) has been derived on the unit circle, we can regard this relationship as valid on the complement of the disc.

Let z_1 and z_2 denote the two roots of the quadratic equation

$$Bz^2 + Fz + \bar{B} = 0.$$

Clearly $|z_1 z_2| = 1$. If $|z_1| = |z_2| = 1$ and $z_1 \neq z_2$, then $z_2 = \bar{z}_1$. Otherwise (assuming $z_1 \neq z_2$) we have $z_2 = \dfrac{1}{\bar{z}_1}$. We can always assume that $|z_2| \geq 1$. Moreover when $|z_1| = |z_2| = 1$, since by hypothesis $\rho \in R$, we have that $\Theta(z_1)$ and $\Theta(z_2)$ exist.

If we now plug z_1 into (24) and z_2 into (25) and solve the two resulting linear operator equations (in the two "unknowns" μ_0 and μ_{-1}) for μ_0, we get that for $z_1 \neq z_2$

$$U(\rho)\mu_0 = V(\rho)\phi_* h_0$$

where

$$U(\rho) := \left[\left[\frac{A - C}{2}\right]\frac{(I - \Theta(z_1)\Theta(1/\bar{z}_2)^*)}{(z_1 - z_2)}\right. \tag{26}$$

$$\left. + \left[\frac{B}{2}\right](I + \Theta(z_1)\Theta(1/\bar{z}_2)^*)\right]$$

$$V(\rho) := \frac{\left[I - \Theta(z_1)\Theta(\frac{1}{\bar{z}_2})^*\right]}{(z_1 - z_2)}. \tag{27}$$

In case $z_1 = z_2$, (26) and (27) take on the following degenerate form:

$$U(\rho) := -C\,\Theta'(z_1) + (\operatorname{sgn} F)\,|B|\,\Theta'(z_1) - B\,\Theta(z_1)$$

$$V(\rho) := -\Theta'(z_1)$$

where sgn F denotes the sign of F. Thus we can conclude from the above that we have that

$$\nu(\rho) = \|U^{-1}(\rho)V(\rho)\|. \tag{28}$$

This is an explicit formula which can be analyzed for any given $T \in C_{\cdot 0}$.

REMARK (3.1) We should note that for $T \in C_0(1)$, (28) agrees

with the formula derived in [3]. Thus for example if T is the compressed shift on $H^2 \ominus mH^2$ where

$$m(z) := \exp\left[\frac{z+1}{z-1}\right]$$

then for $w(z) = (z-1)/2$, one can show from (28) that the discrete spectrum of $w(T)w(T)^*$ is given by the roots of

$$\tan\sqrt{\frac{1}{\rho^2} - 1} + \sqrt{\frac{1}{\rho^2} - 1} = 0$$

in (0, 1). See [2], [3] for some more details about the relevance of this example to the control of delay systems.

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